直交座標(x,y)と極座標 (r,θ) の関係は、

$$x=r\cos\theta$$
$$y=r\sin\theta$$
$$r=(x^2+y^2)^{1/2}$$
$$\theta=\tan^{-1}\left(\frac{y}{x}\right) \qquad (1)$$

である。もし、関数 $f(r,\theta)$ が (r,θ) の関数であれば(1)式から微分の連鎖則を使って、

$$\left(\frac{\partial f}{\partial x}\right)_y=\left(\frac{\partial f}{\partial r}\right)_\theta\left(\frac{\partial r}{\partial x}\right)_y+\left(\frac{\partial f}{\partial \theta}\right)_r\left(\frac{\partial \theta}{\partial x}\right)_y \qquad (2)$$

$$\left(\frac{\partial f}{\partial y}\right)_x=\left(\frac{\partial f}{\partial r}\right)_\theta\left(\frac{\partial r}{\partial y}\right)_x+\left(\frac{\partial f}{\partial \theta}\right)_r\left(\frac{\partial \theta}{\partial y}\right)_x \qquad (3)$$

ここで、

$$\left(\frac{\partial r}{\partial x}\right)_y=\left(\frac{\partial(x^2+y^2)^{1/2}}{\partial x}\right)_y=\frac{x}{(x^2+y^2)^{1/2}}=\cos\theta$$

$$\left(\frac{\partial r}{\partial y}\right)_x=\left(\frac{\partial(x^2+y^2)^{1/2}}{\partial y}\right)_x=\frac{y}{(x^2+y^2)^{1/2}}=\sin\theta$$

$$\left(\frac{\partial \theta}{\partial x}\right)_y=\left(\frac{\partial \tan^{-1}\left(\frac{y}{x}\right)}{\partial x}\right)_y=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r}$$

$$\left(\frac{\partial \theta}{\partial y}\right)_x=\left(\frac{\partial \tan^{-1}\left(\frac{y}{x}\right)}{\partial y}\right)_x=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)=\frac{y}{x^2+y^2}=\frac{\cos\theta}{r}$$

簡単のために、r を一定だとすれば r での微分はゼロになる。これは、円周上の運動の話になる。
(2)式と(3)式を用いれば、r は固定されていて

$$\left(\frac{\partial f}{\partial x}\right)_y=0+\left(\frac{\partial f}{\partial \theta}\right)_r\left(\frac{\partial \theta}{\partial x}\right)_y=-\frac{\sin\theta}{r}\left(\frac{\partial f}{\partial \theta}\right)_r$$

$$\left(\frac{\partial f}{\partial y}\right)_x=0+\left(\frac{\partial f}{\partial \theta}\right)_r\left(\frac{\partial \theta}{\partial y}\right)_x=\frac{\cos\theta}{r}\left(\frac{\partial f}{\partial \theta}\right)_r \qquad (4)$$

ここで、再び、(2)式を適用すれば

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_y=\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)_y\right]=\left[\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)_y\right]_r\left(\frac{\partial \theta}{\partial x}\right)_y$$
$$=\left\{\frac{\partial}{\partial\theta}\left[-\frac{\sin\theta}{r}\left(\frac{\partial f}{\partial\theta}\right)_r\right]\right\}_r\left(-\frac{\sin\theta}{r}\right) \qquad (5)$$
$$=\frac{\sin\theta\cos\theta}{r^2}\left(\frac{\partial f}{\partial\theta}\right)_r+\frac{\sin^2\theta}{r^2}\left(\frac{\partial^2 f}{\partial\theta^2}\right)_r$$

同様にして、

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_x = -\frac{\sin\theta\cos\theta}{r^2}\left(\frac{\partial f}{\partial \theta}\right)_r + \frac{\cos^2\theta}{r^2}\left(\frac{\partial^2 f}{\partial \theta^2}\right)_r \tag{6}$$

それゆえ、

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\sin^2\theta + \cos^2\theta}{r^2}\left(\frac{\partial^2 f}{\partial \theta^2}\right)_r = \frac{1}{r^2}\left(\frac{\partial^2 f}{\partial \theta^2}\right)_r \tag{7}$$

である。